Solution of Algebraic and Transcendental equations

An expression of the form.

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

where all a's are constants provided $a \neq 0$ and 'n' is a positive integer, called polynomial in x of degree 'n'. The polynomial $f(x) = 0$ is called ~~aleg~~ algebraic eqn of degree n.

On the other hand when $f(x)$ is expressed involving some other functions such as trignometric, logarithmic, exponential etc. the $f(x) = 0$ is called transcendental eqn.

$$x^3 + 3x^2 + 4 = 0 \rightarrow \text{algebraic}.$$

$$x^3 + 3x^2 + \cos x + 4 = 0 \rightarrow \text{transendental}$$

Synthetic division by linear factors

Let the division of

$$f_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots a_{n-1} x + a_n$$

by $(x - \alpha)$ where α is any constant. then we have $Q(x)$ as the Quotient and R as the remainder.

then we have.

$$\frac{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n}{(x - \alpha)} = Q(x) + \frac{R}{(x - \alpha)}$$

$\Rightarrow a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = (x-\alpha)\,Q(x) + R$ —— (1)

$\Rightarrow a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$
$= (x-\alpha)\,(b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-1})$
$+ R.$

assume $Q(x) = (b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-1})$

Comparing coeffitients of like powers of 'x' we get

$a_0 = b_0$	$b_0 = a_0$
$a_1 = b_1 - b_0\alpha$	$b_1 = a_1 + b_0\alpha$
$a_2 = b_2 - b_1\alpha$	$\Rightarrow b_2 = a_2 + b_1\alpha$
$\dots$	$\dots$
$a_{n-1} = b_{n-1} - b_{n-2}\alpha$	$b_{n-1} = a_{n-1} + b_{n-2}\alpha$
$a_n = R - b_{n-1}\alpha$	$R = a_n + b_{n-1}\alpha.$

The above results can be expressed in a tabular form known as Horner's Scheme

α	a_0	a_1	$\dots$	a_{n-2}	a_{n-1}	a_n
	b_0	$b_0\alpha$		$b_{n-3}\alpha$	$b_{n-2}\alpha$	$b_{n-1}\alpha$
	$b_0 = a_0$	b_1	$\dots$	b_{n-2}	b_{n-1}	$R = f_n(\alpha)$

Every term in second row is obtained by multiplying the preceeding term in the third row by 'α' whereas every term of third row is obtained by the addition of the corresponding term of first and second row.

Also the remainder

$$R = a_n + b_{n-1}\alpha$$
$$= a_n + (a_{n-1} + b_{n-2}\alpha)\alpha$$
$$= a_n + a_{n-1}\alpha + (a_{n-2} + \alpha b_{n-3})\alpha^2$$
$$= a_n + a_{n-1}\alpha + a_{n-2}\alpha^2 + \dots + a_1\alpha^{n-1} + a_0\alpha^n$$
$$= f_n(\alpha)$$

Note: If the divisor is $x+\alpha$, then we will substitute $-\alpha$ for α and proceed in the same process.

Solution of Equation

The value of 'x' for which an equation $f(x)=0$ is satisfied, is called its root. Geometrically, we can say that the root of an equation $f(x)=0$ is that value of 'x' when the graph of $y=f(x)$ cuts the x-axis.

The process to obtain roots of an equation is called solution of an equation. For $f(x)$ in form of quadratic, cubic, biquadratic equation, algebraic solutions of that equation are available. But for higher degree or transendental equations no direct methods are available. so approximation methods such as Bisection, Secant, Regula-Falsi and Newtonian methods are used.